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Breaking SO(3) into its closed subgroups by Higgs mechanism

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Abstract. A gauged SO(3) symmetry is broken into its closed subgroups by Higgs scalars belonging to the irreducible representations characterized by $j = 2$, 3, 4 and 6. Explicit matrix decompositions of the irreducible representations of SO(3) in terms of the irreducible representations of the closed subgroups are made manifest. Analogous structures between the line defects of liquid crystals and the cosmic strings are notified.

1. Introduction

The closed subgroups of SO(3) are well known to physicists through their applications in crystallography and molecular physics. They are the cyclic groups C_n, dihedral groups D_n, tetrahedral group T, octahedral group O, and the icosahedral group Y. There are also two infinite closed subgroups $C_\infty \approx \mathrm{SO}(2)$ generated by an arbitrary rotation around an axis and D_∞ which is generated by C_∞ and a rotation π around an axis orthogonal to the axis of rotation of C_∞.

Louis Michel, in his remarkable paper [1], has given the list of the little groups of the irreducible representations $j = 0$–6. The little groups of $j = 0$ and $j = 1$ are obviously SO(3) and SO(2), respectively. The others are non-trivial and will be the topic of this paper. It seems that many of these little groups manifest themselves in the phase transitions of liquid crystals [2]. If H is one of the closed subgroups of interest then the conjugacy classes of the homotopy groups π_1 of the coset space $\mathrm{SO}(3)/H$ classify the line defects of the liquid crystals [3]. The homotopy groups satisfy the relation $\pi_1(\mathrm{SO}(3)/H) \approx \pi_1(\mathrm{SU}(2)/H') \approx H'$ if H' is the disconnected double cover of H. The H' is called the binary polyhedral group and they constitute the finite subgroups of SU(2). Therefore the class multiplications of the conjugacy classes of the binary polyhedral groups will be of great importance when two line defects of liquid crystals coalesce [4].

Analogous structures are expected as cosmic strings when SO(3) is taken as a local gauge symmetry [5]. These cosmic strings may arise from the GUT breaking [6] where SO(3) may be embedded as a component of the family symmetry of leptons and quarks. There has been a considerable interest in the finite subgroups of SO(3) or SU(2) to accommodate the family structure of leptons and quarks [7].

Finite subgroups of SU(2) have also been a focus of interest from the mathematical point of view. It is known as the McKay correspondence [8] which associates the columns of the

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character tables of the binary polyhedral groups with the eigenvectors of the incidence matrices (2I-Cartan matrix) of the affine Lie algebras $\hat{A}_n$, $\hat{D}_n \hat{E}(\hat{E}_6, \hat{E}_7, \hat{E}_8)$. Here the correspondence is respectively cyclic groups, dicyclic groups (double covers of the dihedral groups), binary tetrahedral group, binary octahedral group and the binary icosahedral group. Incidence matrices of these affine algebras also play crucial roles in the decomposition of the irreducible representations of SU(2) into the irreducible representations of its finite subgroups [9]. These features of the finite subgroups of SU(2) and the combinations of line defects of liquid crystals as well as the cosmic strings will be the subject of a separate publication [10].

Classification of the little groups of the irreducible representations of compact Lie groups remains an unsolved problem although a great deal of investigation has been made along this line [11]. Even for SO(3) further work is needed to clarify some of the confusion in the literature. In what follows we obtain, in the canonical basis, explicit matrix decompositions of the SO(3) irreps $j = 2, 3, 4, 6$ in terms of the irreps of the closed subgroups of interest. We identify the representation contents of the scalar fields. Assigning the vacuum expectation values to the fields transforming as trivial representations of the closed subgroups we obtain the masses of the gauge bosons of SO(3).

A similar work could have been done by taking the symmetric tensor fields of ranks 2, 3, 4 and 6 with suitable trace conditions [12]. Indeed, this type of approach has been partly discussed in the phase transitions of liquid crystals [13]. In a different paper the connections between these two approaches will be discussed [14].

The paper is organized as follows. In section 2 we discuss the generation relations of the generators of the finite subgroups of SO(3) and SU(2) displaying examples from the two-dimensional irreducible representations of SU(2). In section 3 we discuss the symmetry breaking with the scalars in the irreducible representations $j = 2$ and $j = 3$. Section 4 is devoted to the discussion of the problem for $j = 4$. Section 5 contains the study of the $j = 6$ case where we obtain the dihedral groups $D_n (n = 6, 5, 4, 3, 2)$ as well as the icosahedral group Y as little groups. Finally, we discuss the implications of our approach in physics in the concluding remarks. Matrices generating the irreducible representations $\underline{3}$, $\underline{4}$, $\underline{5}$ of the icosahedral group and the scalar fields of these representations are given in the appendix.

2. Generators and the generation relations of the finite subgroups of SO(3) and SU(2)

The irreducible representations of SO(3) are all real; up to an equivalence, there is only one such representation for each odd dimension $2j + 1$, where j is the angular momentum. An arbitrary group element of SU(2) can be written as $e^{i\boldsymbol{w}\cdot\boldsymbol{J}}$ where $\boldsymbol{J}$ are the usual $(2j + 1) \times (2j + 1)$ matrix representations

$$\begin{aligned} J_{\pm}\phi(jm) &= \sqrt{(j \mp m)(j \pm m + 1)}\phi(jm \pm 1) \\ J_3\phi(jm) &= m\phi(jm) \end{aligned} \tag{1}$$

where $\phi(jm)$ are the basic vectors of the $(2j + 1)$-dimensional irreducible representation of SU(2) when j takes the values $0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$; integer values are for the SO(3) cases. In our calculations $\phi(jm)$ will stand for the scalar fields associated with the related irreducible representations.

The finite subgroups of SU(2) can be generated by discrete rotations around the properly chosen two axes. Let A and B denote such rotations. Then the generation relations of the finite subgroups of SU(2) and SO(3) are given by

$$A^p = B^q = C^r = Z \tag{2}$$

where $C = AB$, $Z^2 = 1$ for SU(2) and $Z = 1$ for SO(3). We will also use occasionally the notation of [15] for the designation of these groups; the polyhedral subgroups of SO(3) are denoted by (pqr) and the notation $\langle pqr\rangle$ will stand for the binary polyhedral subgroups of SU(2). Here (pqr) takes the values $(nn1)$ for the cyclic groups, $(n22)$ for the dihedral groups, while (332), (432) and (532) stand for the tetrahedral, octahedral and icosahedral subgroups of SO(3), respectively. Here n is an integer $n \geqslant 1$. Their double covers are the binary polyhedral groups $\langle pqr\rangle$. There are also two infinite subgroups of SO(3), $C_\infty \approx \mathrm{SO}(2)$ generated by an arbitrary rotation around an axis, say the third axis, $\mathrm{e}^{\mathrm{i}\theta J_3}$; and the infinite dihedral group whose generators, in this paper, will be taken to be $A = \mathrm{e}^{\mathrm{i}\theta J_3}$ and $B = \mathrm{e}^{\mathrm{i}\pi J_1}$. The generation relation

$$BAB^{-1} = A^{-1} \qquad B^2 = 1 \tag{3}$$

can be used for all dihedral groups including D_∞.

In general, a rotation by an angle θ around an arbitrary axis $Q = \eta_i J_i$, where $|\eta| = 1$, can be written as $\mathrm{e}^{\mathrm{i}\theta Q}$. For example, the generators

$$A = \mathrm{e}^{\mathrm{i}(2\pi/n)J_3} \qquad B = \mathrm{e}^{\mathrm{i}\pi J_1} \tag{4}$$

generate the dihedral group D_n, $(n22)$, of order $2n$, if j is an integer, otherwise they generate the dicyclic group $\langle n22\rangle$ of order $4n$.

Below we list the generators and the generation relations of the polyhedral subgroups of SO(3):

Finite subgroups of SO(3)	Generators	Generation relations
C_n:$(nn1)$	$A = \exp \mathrm{i}\frac{2\pi}{n} J_3$	$A^n = 1$
D_n:$(n22)$	$A = \exp \frac{\mathrm{i}2\pi}{n} J_3$, $B = \exp \mathrm{i}\pi J_1$	$A^n = B^2 = C^2 = 1$
T:(332)	$A = \exp \frac{\mathrm{i}2\pi}{3} \frac{1}{\sqrt{3}}(J_1 + J_2 + J_3)$	$A^3 = B^3 = (AB)^2 = 1$, $B = A^2C$
	$C = \exp \mathrm{i}\pi J_1$	
O : (432)	$A = \exp \frac{\mathrm{i}2\pi}{4} J_3$	$A^4 = B^3 = (AB)^2 = 1$
	$B = \exp \frac{\mathrm{i}2\pi}{3} \frac{1}{\sqrt{3}}(J_1 + J_2 + J_3)$	
Y:(532)	$A = \exp \frac{\mathrm{i}2\pi}{5} \frac{\sigma J_1 + J_3}{\sqrt{2+\sigma}}$	$A^5 = B^3 = C^2 = 1$
	$B = \exp \frac{\mathrm{i}2\pi}{3}(-\sigma J_2 + \tau J_3)$	
	$\tau = \frac{1}{2}(1+\sqrt{5})$	
	$\sigma = \frac{1}{2}(1-\sqrt{5})$	

(5)

Let us further illustrate this point giving an example from the familiar two-dimensional representation of SU(2). The Lie algebra of SU(2) is represented by $J_i = \sigma_i/2$ where $\sigma_i (i = 1, 2, 3)$ are the usual Pauli matrices. Consider the rotations

$$A = \mathrm{e}^{\mathrm{i}\frac{2}{3}\pi Q}, C = \mathrm{e}^{\mathrm{i}\pi\frac{1}{2}\sigma_1} \tag{6}$$

where $Q = \frac{1}{2\sqrt{3}}(\sigma_1 + \sigma_2 + \sigma_3)$ with $Q^2 = \frac{1}{4}$.

They are given by the matrices

$$A = \tfrac{1}{2}(1 + \mathrm{i}\sigma_1 + \mathrm{i}\sigma_2 + \mathrm{i}\sigma_3) \qquad C = \mathrm{i}\sigma_1 \tag{7}$$

where $A^3 = -1$, $C^2 = -1$, $B = A\dagger C = \frac{1}{2}(1 + \mathrm{i}\sigma_1 + \mathrm{i}\sigma_2 - \mathrm{i}\sigma_3)$, $B^3 = -1$.

Therefore the three generators, A, B, C, satisfy the generation relations $A^3 = B^3 = (AB)^2 = -1$ as stated in (2) and generate the binary tetrahedral group $\langle 332\rangle$ of order 24. The elements of the group are given by the 2×2 unitary matrices.

$$\pm 1, \pm\mathrm{i}\sigma_1, \pm\mathrm{i}\sigma_2, \pm\mathrm{i}\sigma_3, \tfrac{1}{2}(\pm 1 \pm \mathrm{i}\sigma_1 \pm \mathrm{i}\sigma_2 \pm \mathrm{i}\sigma_3). \tag{8}$$

A 3×3 matrix representation of the tetrahedral group of order 12 is either obtained by substituting the 3×3 matrix representations of J_i in (5) or taking the inner automorphism of the binary tetrahedral group in (8).

3. Symmetry breaking with the Higgs scalars in the representations $j = 2$ and $j = 3$

The standard Lagrangian of a local gauge theory without fermions is given by

$$L = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}(D_\mu\phi)^\dagger(D_\mu\phi) - V(\phi) \tag{9}$$

where the field strengths $F_{\mu\nu}$ and the covariant derivative D_μ are given by

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu W_\nu - \partial_\nu W_\mu + gW_\mu \times W_\nu \\ D_\mu &= \partial_\mu - \mathrm{i}gW_\mu, \;\; W_\mu = \boldsymbol{J}\cdot\boldsymbol{W}_\mu \end{aligned} \tag{10}$$

with $\boldsymbol{J}$ being the $(2j+1)\times(2j+1)$ matrix representation obtained by the expression (1). A general Higgs potential restricted by renormalization can be written as

$$V(\chi) = \alpha + \beta\chi_a\chi_a + \gamma f_{abc}\chi_a\chi_b\chi_c + \delta g_{abcd}\chi_a\chi_b\chi_c\chi_d \tag{11}$$

where $\chi_a (a = 1, 2, \ldots, 2j+1)$ are real scalar fields which can be defined in terms of the complex scalar fields $\phi(jm)$. The tensors f_{abc} and g_{abcd} should be chosen appropriately for each irreducible representation of degree $2j+1$. The minimum of the Higgs potential of (11) can be obtained by assigning the vacuum expectation values to the Higgs scalars $\langle\chi_a\rangle = v_a$. The details of the symmetry breaking patterns are very important which we will discuss for the irreducible representation $j = 2$. The main objective of this paper is then to identify the non-zero expectation values associated with the appropriate little groups of SO(3). We start with the representations of $\boldsymbol{J}$ given in (1) where $\phi^*(jm) = (-1)^m\phi(j-m)$ (the same relation satisfied by the spherical harmonics) and transform it to a real basis of field which we denoted by χ.

(a) $j = 2$ *case*

This irreducible representation has two little groups D_∞ and D_2. In the $\phi(2m)$ basis $-2 \leqslant m \leqslant 2$ the generators of D_∞ read

$$A = \exp \mathrm{i}wJ_3 = \begin{bmatrix} \mathrm{e}^{2iw} & & & & \\ & \mathrm{e}^{\mathrm{i}w} & & & \\ & & 1 & & \\ & & & \mathrm{e}^{-\mathrm{i}w} & \\ & & & & \mathrm{e}^{-2\mathrm{i}w} \end{bmatrix}$$

$$B = \exp \mathrm{i}\pi J_1 = \begin{bmatrix} & & & & 1 \\ & & & 1 & \\ & & 1 & & \\ & 1 & & & \\ 1 & & & & \end{bmatrix}. \tag{12}$$

Choosing the real field $\chi = T\phi$ as

$$\begin{aligned} \chi_1 &= \tfrac{1}{\sqrt{2}}(\phi(22) + \phi(2-2)) \qquad & \chi_3 &= \tfrac{1}{\sqrt{2}}(\phi(21) - \phi(2-1)) \\ \chi_2 &= \tfrac{1}{\sqrt{2}}\,\mathrm{i}(\phi(22) - \phi(2-2)) \qquad & \chi_4 &= \tfrac{1}{\sqrt{2}}\,\mathrm{i}(\phi(21) + \phi(2-1)) \\ \chi_5 &= \phi(20) & & \end{aligned} \tag{13}$$

and transforming the generators A and B to the basis of the χ fields one gets

$$A'(w) = TAT^{\dagger} = \begin{bmatrix} a(2w) & & \\ & a(w) & \\ & & 1 \end{bmatrix} \qquad B' = TAT^{\dagger} = \begin{bmatrix} \sigma_3 & & \\ & -\sigma_3 & \\ & & 1 \end{bmatrix} \tag{14}$$

where

$$a(nw) = \begin{bmatrix} \cos nw & \sin nw \\ -\sin nw & \cos nw \end{bmatrix} \qquad n = 0, 1, 2, \ldots$$

and

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is the third Pauli matrix.

It is clear from (14) that $A'(w)$ and B' are in the block diagonal forms where each block represents an irreducible representation of D_∞. The group D_∞ has two irreducible representations of degree 1 (one is the trivial representation) and an infinite number of two-dimensional irreducible representations characterized by $[m]$ [16]. From $A'(w)$ and B' we conclude that $\phi(20)$ transforms as a trivial representation while the fields $(\phi(22), \phi(2-2))$ and $(\phi(21), \phi(2-1))$ transform as doublets of [2] and [1], respectively. By assigning the vacuum expectation values to χ_5, $\langle\chi_5\rangle = \langle\phi(20)\rangle = v_5$ the SO(3) symmetry is broken to D_∞ where (W_1, W_2) or $W^\pm$ gain equal masses while W_3 remains massless. The massive gauge bosons $W^\pm$ transform as a doublet of [1] of D_∞ generated by the matrices $a(w)$ and $-\sigma_3$ while W_3 belongs to the non-trivial representation of degree 1.

If we let $w = \pi$, $A'(w)$ takes the form

$$A'(\pi) = \begin{bmatrix} I & & \\ & -I & \\ & & 1 \end{bmatrix} \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{15}$$

Now $A'(\pi)$ and B' generate the reducible representation of degree 5 of D_2. One can immediately note that the fields χ_1 and χ_5 transform as the trivial representation of D_2. Assigning the vacuum expectation values to the fields $\langle\chi_1\rangle = v_1 = \sqrt{2}\langle\phi(22)\rangle = \sqrt{2}\langle\phi(2-2)\rangle$ and $\langle\phi(20)\rangle = v_5$ one can break the SO(3) symmetry to the subgroup D_2 of D_∞. The gauge bosons W_1, W_2 and W_3 gain the masses

$$m_1^2 = g^2(v_1 + \sqrt{3}v_5)^2 \qquad m_2^2 = g^2(v_1 - \sqrt{3}v_5)^2 \qquad m_3^2 = 4g^2v_1^2 \tag{16}$$

and each transform as one of the non-trivial representations of degree 1 of D_2.

Now we would like to discuss the symmetry breaking patterns using the general potential $V(\chi)$ given by (11). In almost all papers in the literature [17] the breaking SO(3) to D_∞ or D_2 is made by the use of a second rank symmetric traceless tensor $T_{ij} = T_{ji}$, $T_{ii} = 0$.

It is trivial to make one-to-one correspondence between the fields $\chi_a (a = 1, \ldots, 5)$ and the components of the tensor T_{ij}[14]:

$$\begin{aligned} &T_{11} = \tfrac{1}{\sqrt{2}}\chi_1 + \tfrac{1}{\sqrt{6}}\chi_5 \qquad T_{22} = \tfrac{-1}{\sqrt{2}}\chi_1 + \tfrac{1}{\sqrt{6}}\chi_5 \qquad T_{33} = \tfrac{-2}{\sqrt{6}}\chi_5 \\ &T_{12} = \tfrac{1}{\sqrt{2}}\chi_2 \qquad T_{13} = \tfrac{1}{\sqrt{2}}\chi_3 \qquad T_{23} = \tfrac{1}{\sqrt{2}}\chi_4. \end{aligned} \tag{17}$$

The potential of (11) in terms of the field T_{ij} would read

$$V(T) = \alpha + \beta\,\mathrm{Tr}\,T^2 + \gamma\,\mathrm{Tr}\,T^3 + \delta[\mathrm{Tr}\,T^2]^2. \tag{18}$$

Symmetry-breaking patterns of this potential have been discussed in condensed matter physics associated with the phase transitions of the uniaxial and biaxial nematic liquid crystals. T_{ij} is a symmetric traceless 3×3 matrix which can be transformed to the diagonal form where the eigenvalues T_1, T_2 and T_3 can be related to the vacuum expectation values v_1 and v_5 by the relations

$$\begin{aligned} T_1 &= \tfrac{1}{\sqrt{2}} v_1 + \tfrac{1}{\sqrt{6}} v_5 \\ T_2 &= \tfrac{-1}{\sqrt{2}} v_1 + \tfrac{1}{\sqrt{6}} v_5 \\ T_3 &= \tfrac{-2}{\sqrt{6}} v_5. \end{aligned} \tag{19}$$

Now the potential (18) reads

$$V(v_1, v_5) = \alpha + \beta(v_1^2 + v_5^2) + \tfrac{1}{\sqrt{6}} \gamma v_5 (3v_1^2 - v_5^2) + \delta(v_1^2 + v_5^2)^2. \tag{20}$$

We note that for a symmetric 3×3 real matrix $[\mathrm{Tr}\, T^3]^2 \leqslant \frac{1}{6}[\mathrm{Tr}\, T^2]^3$ is satisfied. This results in the relation $3v_5^2 \geqslant v_1^2$ or $3v_5^2 < v_1^2$.

The relations $\partial V/\partial v_1 = \partial V/\partial v_5 = 0$ lead to the conditions

(i) $\gamma = 0$, $v_1^2 + v_5^2 = -\beta/2\gamma$

(ii) $\gamma \neq 0$, either $v_1 = 0$, $v_5 \neq 0$ or $3v_5^2 = v_1^2 \neq 0$.

For the latter possiblities we should have $\beta < 0, \gamma < 0$ and $\delta > 0$ to satisfy the local minimality. They correspond to SO(3)→ D_∞ breaking where one of the gauge bosons remains massless. In terms of the eigenvalues of the matrix T, they represent the degenerate case T: $(1, 1, -2)v_5/\sqrt{6}$ for $v_1 = 0$. For $v_1 = \pm\sqrt{3}v_5$ one still gets the degenerate eigenvalues but in different orderings. In condensed matter physics this corresponds to a phase transition from the isotropic liquid to uniaxial nematic liquid crystal.

One can work out the most general case where $v_1 \neq 0$ and $v_5 \neq 0$ with $3v_5^2 \neq v_1^2$ and show that the minimum of the potential is obtained only for $\gamma = 0, \beta < 0$ and $\delta > 0$. For simplicity one can study the case $v_5 = 0$ and $v_1^2 = -\beta/2\delta$ and show that the minimum of the potential is achieved for $\beta < 0$ and $\delta > 0$. This choice breaks SO(3) to D_2 where T has non-degenerate eigenvalues T : $(1, -1, 0)2v_5/\sqrt{6}$. The alternative choices $v_5 = \pm\sqrt{3}v_1$ are required by the minimality of the potential which corresponds to reshuffling of the non-degenerate eigenvalues as $(1, 0, -1)$ or $(0, 1, -1)$.

This breaking, in condensed matter physics, represents a phase transition from the isotropic liquid to a biaxial nematic liquid crystal. If we choose $v_1 = 0$ and $v_5^2 = -\beta/2\delta$ then we obtain SO(3)→ D_∞ breaking provided $\beta < 0$ and $\delta > 0$.

(b) *The little groups of the irreducible representation $j = 3$*

By certain non-zero components of the vacuum expectation values for $\langle\phi\rangle$ one can break SO(3) to C_∞, D_3 and the tetrahedral group T. To achieve this, let $A = \exp \mathrm{i}wJ_3$ be the generator of C_∞. To generate D_∞ and thereof its subgroup D_3 we take $B = \exp \pi \mathrm{i} J_1$ as usual. The real basis of fields are defined by

$$\begin{aligned} \chi_1 &= \tfrac{1}{\sqrt{2}}(\phi(33) - \phi(3-3)) & \chi_5 &= \tfrac{1}{\sqrt{2}}(\phi(31) - \phi(3-1)) \\ \chi_2 &= \tfrac{1}{\sqrt{2}}\,\mathrm{i}(\phi(33) + \phi(3-3)) & \chi_6 &= \tfrac{1}{\sqrt{2}}\,\mathrm{i}(\phi(31) + \phi(3-1)) \\ \chi_3 &= \tfrac{1}{\sqrt{2}}(\phi(32) + \phi(3-2)) & \chi_7 &= \phi(30) \\ \chi_4 &= \tfrac{1}{\sqrt{2}}\,\mathrm{i}(\phi(32) - \phi(3-2)). \end{aligned} \tag{21}$$

In the χ-basis the generators of D_∞ are in the block diagonal forms:

$$A'(w) = \begin{bmatrix} a(3w) & & & \\ & a(2w) & & \\ & & a(w) & \\ & & & 1 \end{bmatrix} \qquad B' = \begin{bmatrix} \sigma_3 & & & \\ & -\sigma_3 & & \\ & & \sigma_3 & \\ & & & -1 \end{bmatrix}. \tag{22}$$

A glance at the matrices $A'(w)$ and B' shows that D_∞ is not a little group of $j = 3$ for D_∞ does not possess a trivial representation whereas its subgroup C_∞ has one. Therefore, the non-zero vacuum expectation value $\langle\phi(30)\rangle = v_7$ breaks SO(3) to C_∞.

Letting $w = \pi$ we note that the field χ_4 transforms as a trivial representation of D_2. Nevertheless, this is not a little group of $j = 3$ because, as we will see in the case of tetrahedral group T, the field χ_4 serves as the only trivial representation of T where D_2 is a subgroup. Due to a lemma of [1] the breaking occurs only for the maximal group preserving the same trivial representation.

To obtain the generators of D_3 we let $w = 2\pi/3$ in $A'(w)$ of (18). Then $A'(2\pi/3)$ takes the form

$$A'\left(\frac{2\pi}{3}\right) = \begin{bmatrix} I & & & \\ & R & & \\ & & R^2 & \\ & & & 1 \end{bmatrix} \qquad R = \frac{1}{2}\begin{bmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix} \qquad R^3 = 1. \tag{23}$$

Using $A'(2\pi/3)$ and B' together we deduce that the field χ_1 transforms as a trivial representation. The other fields χ_2 and χ_7 remain to be the same non-trivial singlet, the pair of fields (χ_3, χ_4) and (χ_5, χ_6) transform as the doublet of D_3. When $\langle\chi_1\rangle = v_1$ takes the non-zero vacuum expectation value the gauge bosons gain the masses $m_{W^\pm} = \sqrt{3}v_1$ and $m_{W_3} = 3\sqrt{2}v_1$ which leads to the relation $m_{W_3} = \sqrt{6}m_{W^\pm}$. After the symmetry breakdown $W^\pm$ transform as a doublet while W_3 transform as a non-trivial representation of degree 1.

The group D_3 could have been generated by $A(2\pi/3)$ and $C = \exp \mathrm{i}\pi J_2$. This would be a different embedding of D_3 in SO(3) and would yield the field χ_2 transform as a trivial representation. This indicates that the fields χ_1 and χ_2 change their roles as we shift from a rotation of π around the x_1-axis to x_2-axis. The generators $A(2\pi/3)$, B and C together generate a larger group D_6 which does not possess a trivial representation at all.

Now we discuss the breaking of SO(3) to the tetrahedral group $T = (332)$ with the scalars of $j = 3$. The tetrahedral group is isomorphic to the group A_4 of even permutations of four letters. The generators of the tetrahedral group of order 12 can be taken to be

$$A = \exp \tfrac{2}{3}\mathrm{i}\pi Q \qquad Q = \tfrac{1}{\sqrt{3}}(J_1 + J_2 + J_3)$$

and

$$B = \exp \mathrm{i}\pi J_1. \tag{24}$$

Instead of B one can also take $D = \exp \mathrm{i}\pi J_3$ which is already diagonal. The matrix A is a 7×7 unitary matrix in the $\phi(3m)$ basis and does not take block diagonal form if we pass over to the real basis of (21). However, it is not difficult to transform the matrices $\exp \mathrm{i}(2\pi/3)Q$ and $\exp \mathrm{i}\pi J_i$ $(i = 1, 2, 3)$ to the block diagonal forms where one can easily read the representation contents of the χ or ϕ fields. For this purpose we introduce the fields

$$\begin{aligned} &\eta_1 = \chi_3 \qquad \eta_4 = \chi_7 \\ &\eta_2 = \tfrac{\sqrt{6}}{4}\chi_2 - \tfrac{\sqrt{10}}{4}\chi_6 \qquad \eta_5 = \tfrac{\sqrt{10}}{4}\chi_2 + \tfrac{\sqrt{6}}{4}\chi_6 \\ &\eta_3 = \tfrac{\sqrt{6}}{4}\chi_1 + \tfrac{\sqrt{10}}{4}\chi_5 \qquad \eta_6 = -\tfrac{\sqrt{10}}{4}\chi_1 + \tfrac{\sqrt{6}}{4}\chi_5 \\ &\eta_7 = \chi_4. \end{aligned} \tag{25}$$

In this new basis of the fields the matrices A and D take the block diagonal forms

$$A' = \begin{bmatrix} E & & \\ & E & \\ & & 1 \end{bmatrix} \qquad D' = \begin{bmatrix} F & & \\ & F & \\ & & 1 \end{bmatrix} \tag{26}$$

where

$$E = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

We note that the generation relation is trivial

$$A'^3 = G'^3 = D'^2 = 1 \tag{27}$$

where G' is defined through the relation

$$A'G' = D' \qquad G' = A'^2 D'. \tag{28}$$

The matrices $\exp \mathrm{i}\pi J_i$ $(i = 1, 2, 3)$ generate the dihedral subgroup D_2 of the tetrahedral group T. It is obvious from (22) that the fields (η_1, η_2, η_3) and (η_4, η_5, η_6) transform as the triplet representation of the tetrahedral group. When $\eta_7 = \chi_4$ takes the non-zero vacuum expectation value $\langle\chi_4\rangle = v_4 = -\mathrm{i}\sqrt{2}\langle\phi(32)\rangle = \mathrm{i}\sqrt{2}\langle\phi(3-2)\rangle$, SO(3) is broken to its tetrahedral subgroup. The gauge bosons gain equal masses of $m_{W^\pm} = m_{W_3} = \sqrt{2}v_4$. The gauge bosons also transform as a triplet under the tetrahedral group so that either triplet of the scalar fields or a mixture of them can be gauged away to give the gauge bosons the longitudinal polarization.

We should also note the cyclic little groups of $j = 3$ which are C_3, C_2 and C_1. They are respectively obtained when we assign $w = 2\pi/3$, $2\pi/2$, and 2π in $A'(w)$ of (18). The fields χ_1, χ_2 and χ_7 transform as the trivial representation of C_3. Similarly the fields χ_3, χ_4 and χ_7 transform as the trivial representation of C_2. For C_1 all fields are in the trivial representation.

4. Breaking with the scalars of *j* = 4

Here we discuss the breaking of SO(3) to all possible little groups of the irreducible representation $j = 4$. They are the dihedral groups D_∞, $D_n (n = 2, 3, 4)$, the cyclic groups C_2, C_1, and the octahedral group O.

(a) *The dihedral groups*
To discuss the dihedral groups we first write down the generators of D_∞ in the $\phi(4m)$, $-4 \leqslant m \leqslant 4$ basis:

$$A = \begin{bmatrix} e^{4\mathrm{i}w} &&&&&&&& \\ & e^{3\mathrm{i}w} &&&&&&& \\ && e^{2\mathrm{i}w} &&&&&& \\ &&& e^{\mathrm{i}w} &&&&& \\ &&&& 1 &&&& \\ &&&&& e^{-\mathrm{i}w} &&& \\ &&&&&& e^{-2\mathrm{i}w} && \\ &&&&&&& e^{-3\mathrm{i}w} & \\ &&&&&&&& e^{-4\mathrm{i}w} \end{bmatrix}$$

$$B = \begin{bmatrix} & & & & & & & & 1 \\ & & & & & & & 1 & \\ & & & & & & 1 & & \\ & & & & & 1 & & & \\ & & & & 1 & & & & \\ & & & 1 & & & & & \\ & & 1 & & & & & & \\ & 1 & & & & & & & \\ 1 & & & & & & & & \end{bmatrix}. \tag{29}$$

When we pass over to the real basis $\chi = T\phi$ with

$$\begin{aligned} \chi_1 &= \tfrac{1}{\sqrt{2}}(\phi(44) + \phi(4-4)) & \chi_5 &= \tfrac{1}{\sqrt{2}}(\phi(42) + \phi(4-2)) \\ \chi_2 &= \tfrac{1}{\sqrt{2}}\,\mathrm{i}(\phi(44) - \phi(4-4)) & \chi_6 &= \tfrac{1}{\sqrt{2}}\,\mathrm{i}(\phi(42) - \phi(4-2)) \\ \chi_3 &= \tfrac{1}{\sqrt{2}}(\phi(43) - \phi(4-3)) & \chi_7 &= \tfrac{1}{\sqrt{2}}(\phi(41) - \phi(4-1)) \\ \chi_4 &= \tfrac{1}{\sqrt{2}}\,\mathrm{i}(\phi(43) + \phi(4-3)) & \chi_8 &= \tfrac{1}{\sqrt{2}}\,\mathrm{i}(\phi(41) + \phi(4-1)) \\ \chi_9 &= \phi(40) \end{aligned} \tag{30}$$

the matrices A and B read

$$A'(w) = \begin{bmatrix} a(4w) & & & & \\ & a(3w) & & & \\ & & a(2w) & & \\ & & & a(w) & \\ & & & & 1 \end{bmatrix} \qquad B' = \begin{bmatrix} \sigma_3 & & & & \\ & -\sigma_3 & & & \\ & & \sigma_3 & & \\ & & & -\sigma_3 & \\ & & & & 1 \end{bmatrix} \tag{31}$$

where $a(nw)$ and σ_3 are defined in (14). We conclude from (31) that the nine-dimensional irreducible representation of SO(3) branches as $\underline{9} = \underline{1} + [1] + [2] + [3] + [4]$ in terms of the irreducible representations of D_∞. When $\phi(40)$ is assigned to the vacuum expectation value $\langle\phi(40)\rangle = v_4$, then SO(3) is broken to D_∞, where $W^\pm$ gain masses but W_3 remain massless.

The subgroups D_2, D_3 and D_4 of D_∞ in the $j = 4$ representation can be obtained by letting w take respectively $2\pi/2$, $2\pi/3$ and $2\pi/4$ in the generator $A'(w)$ of (31). If we set $w = \pi$, $A'(\pi)$ turns out to be a diagonal matrix with elements ± 1. It is obvious to see then that the fields χ_1, χ_5 and χ_9 transform as the trivial representation of D_2; when assigned vacuum expectation values the SO(3) is broken to D_2. In the case of $w = 2\pi/3$ we obtain

$$A'\left(\frac{2\pi}{3}\right) = \begin{bmatrix} R^2 & & & & \\ & I & & & \\ & & R & & \\ & & & R^2 & \\ & & & & 1 \end{bmatrix}. \tag{32}$$

The matrices $A'(2\pi/3)$ and the B' in (31) generate the dihedral group D_3 from which we read that χ_4 and χ_9 transform as the trivial representation and break SO(3) to D_3 when assigned the non-zero vacuum expectation values. The representation properties of the remaining fields are also obvious. As we have noted earlier $A(2\pi/3)$ and $\exp \mathrm{i}\pi J_2$ would also generate another dihedral group D_3 in SO(3). If this is the case then the scalars χ_3 and χ_9 would turn out to be the trivial representation. It is not difficult to check that $\exp \mathrm{i}(2\pi/3)J_3$, $\exp \mathrm{i}\pi J_1$ and $\exp \mathrm{i}\pi J_2$ generate the larger group D_6 which is not a little group of the irreducible representation $j = 4$.

Letting $w = \pi/2$ in $A'(w)$ leads to the matrix

$$A'\left(\frac{\pi}{2}\right) = \begin{bmatrix} I & & & & \\ & -\mathrm{i}\sigma_2 & & & \\ & & -I & & \\ & & & \mathrm{i}\sigma_2 & \\ & & & & 1 \end{bmatrix}. \tag{33}$$

The matrices B' of (31) and $A'(\pi/2)$ of (33) generate the dihedral group D_4 of order 8. The trivials of D_4 are the scalars χ_1 and χ_9. The fields χ_2, χ_5 and χ_6 transform as representations of degree 1 while (χ_3, χ_4) and (χ_7, χ_8) transform as doublets. If we try to generate D_4 with $A'(\pi/2)$ and $C = \exp \mathrm{i}\pi J_2$ we get the same group as before and no alternative breaking occurs. The gauge bosons $W^{\pm}$ transform as a D_4 doublet while W_3 belongs to one of its singlet. Furthermore, if we assume $\langle\chi_1\rangle \neq 0$ but $\langle\chi_9\rangle = 0$ one can also predict a mass relation $m_{W_3} = 2\sqrt{2} m_{W^{\pm}}$.

(b) *The octahedral group O*

The irreducible representation $j = 4$ is the minimal dimension by which one can break SO(3) to the octahedral group O. From (5) we see that the octahedral group is generated by

$$A = \exp \mathrm{i}\frac{\pi}{2} J_3 \qquad B = \exp \mathrm{i}\frac{2\pi}{3}\frac{1}{\sqrt{3}}(J_1 + J_2 + J_3) \tag{34}$$

where J_is are the 9×9 matrix representation of (1). These matrices cannot be put simultaneously into the block diagonal forms in the χ basis in contrast to the cases of dihedral groups. However, a further transformation $\eta = S\chi$ with

$$\begin{aligned} &\eta_1 = \sqrt{\tfrac{5}{12}}\chi_1 + \sqrt{\tfrac{7}{12}}\chi_9 \qquad \eta_4 = \chi_2 \\ &\eta_2 = \sqrt{\tfrac{7}{12}}\chi_1 - \sqrt{\tfrac{5}{12}}\chi_9 \qquad \eta_5 = \tfrac{\sqrt{2}}{4}\chi_3 - \tfrac{\sqrt{14}}{4}\chi_7 \\ &\eta_3 = \chi_5 \qquad \eta_6 = \tfrac{\sqrt{2}}{4}\chi_4 + \tfrac{\sqrt{14}}{4}\chi_8 \qquad \eta_7 = \chi_6 \\ &\eta_8 = \tfrac{\sqrt{14}}{4}\chi_3 + \tfrac{\sqrt{2}}{4}\chi_7 \\ &\eta_9 = -\tfrac{\sqrt{14}}{4}\chi_4 + \tfrac{\sqrt{2}}{4}\chi_8 \end{aligned} \tag{35}$$

transform the matrices of (34) to the block diagonal forms.

$$A' = \begin{bmatrix} 1 & & & \\ & \sigma_3 & & \\ & & K & \\ & & & -K \end{bmatrix} \qquad B' = \begin{bmatrix} 1 & & & \\ & R^2 & & \\ & & E^2 & \\ & & & E^2 \end{bmatrix} \tag{36}$$

where

$$K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

and σ_3, R and E are defined earlier. It is certain that the generation relation $A'^4 = B'^3 = (A'B')^2 = 1$ is satisfied.

The octahedral group is isomorphic to the symmetric group S_4 which has two representations of degree 1 (one is the trivial representation), one representation of degree 2 and two inequivalent representations of degree 3. The decomposition in (36) indicates that the nine-dimensional irreducible representation of SO(3) branches as $\underline{9} = \underline{1} + \underline{2} + \underline{3} + \underline{3}'$ in terms of the irreducible representation of S_4. Thus the field of η_1 transform as a trivial representation while (η_2, η_3) as a doublet, (η_4, η_5, η_6), and (η_7, η_8, η_9) as two inequivalent triplet representations. We wish to emphasize that the fields (η_4, η_5, η_6) transform just like the gauge fields (W_1, W_2, W_3) do under S_4. This indicates that in the symmetry breakdown $SO(3) \longrightarrow S_4$ the fields (η_4, η_5, η_6) are gauged away to yield the longitudinal components of W bosons. The gauge bosons gain the equal masses $m_W = \sqrt{20/3}v_1$ through the vacuum expectation values.

$$\begin{aligned} \langle\phi(44)\rangle &= \langle\phi(4-4)\rangle = \sqrt{\tfrac{5}{24}}\langle\eta_1\rangle = \sqrt{\tfrac{5}{24}}v_1 \\ \langle\phi(40)\rangle &= \sqrt{\tfrac{7}{12}}\langle\eta_1\rangle = \sqrt{\tfrac{7}{12}}v_1. \end{aligned} \tag{37}$$

It is also obvious from (31) that for $w = \pi$ the $A'(\pi)$ generate the cyclic group C_2 whose trivial representation corresponds to the fields χ_1, χ_2, χ_5, χ_6 and χ_9. Therefore, C_2 is also a little group as well as C_1 which corresponds to the unit matrix.

5. Breaking by the scalars in *j* = 6

This is the minimal dimension where one can break SO(3) to the icosahedral group Y which is isomorphic to the group of even permutations A_5 of five letters. Among the possible little groups are D_∞, Y, O, T, $D_n(n = 6, 5, 4, 3, 2)$ and $C_m(m = 3, 2, 1)$. We will not consider the little groups O and T in detail in this section as they have been treated in earlier sections. Again we start by writing the generators of D_∞ and discussing all the little groups D_n and C_m types as the subgroups of D_∞. The case of the icosahedral group will require some detailed considerations.

(a) *The dihedral groups*

In the $\phi(6m)$, $-6 \leqslant m \leqslant 6$, basis we take as generators the $A(w) = \exp \mathrm{i}wJ_3$ which is the 13×13 diagonal matrix and $B = \exp \mathrm{i}\pi J_1$ which is the usual off-diagonal matrix with entries 1. Similar to the previous cases we define the real scalars by

$$\begin{aligned} \chi_1 &= \tfrac{1}{\sqrt{2}}(\phi(66) + \phi(6-6)) & \chi_7 &= \tfrac{1}{\sqrt{2}}(\phi(63) - \phi(6-3)) \\ \chi_2 &= \tfrac{1}{\sqrt{2}}\,\mathrm{i}(\phi(66) - \phi(6-6)) & \chi_8 &= \tfrac{1}{\sqrt{2}}\,\mathrm{i}(\phi(63) + \phi(6-3)) \\ \chi_3 &= \tfrac{1}{\sqrt{2}}(\phi(65) - \phi(6-5)) & \chi_9 &= \tfrac{1}{\sqrt{2}}(\phi(62) + \phi(6-2)) \\ \chi_4 &= \tfrac{1}{\sqrt{2}}\,\mathrm{i}(\phi(65) + \phi(6-5)) & \chi_{10} &= \tfrac{1}{\sqrt{2}}\,\mathrm{i}(\phi(62) - \phi(6-2)) \\ \chi_5 &= \tfrac{1}{\sqrt{2}}(\phi(64) + \phi(6-4)) & \chi_{11} & \tfrac{1}{\sqrt{2}}(\phi(61) - \phi(6-1)) \\ \chi_6 &= \tfrac{1}{\sqrt{2}}\,\mathrm{i}(\phi(64) - \phi(6-4)) & \chi_{12} &= \tfrac{1}{\sqrt{2}}\,\mathrm{i}(\phi(61) + \phi(6-1)) \\ \chi_{13} &= \phi(60). \end{aligned} \tag{38}$$

In the real basis of χ the generators of D_∞ would read

$$A'(w) = \begin{bmatrix} a(6w) & & & & & & \\ & a(5w) & & & & & \\ & & a(4w) & & & & \\ & & & a(3w) & & & \\ & & & & a(2w) & & \\ & & & & & a(w) & \\ & & & & & & 1 \end{bmatrix} \tag{39a}$$

$$B' = \begin{bmatrix} \sigma_3 & & & & & & \\ & -\sigma_3 & & & & & \\ & & \sigma_3 & & & & \\ & & & -\sigma_3 & & & \\ & & & & \sigma_3 & & \\ & & & & & -\sigma_3 & \\ & & & & & & 1 \end{bmatrix}. \tag{39b}$$

The vacuum expectation values $\langle\phi(60)\rangle = v_{13}$ breaks the symmetry to D_∞. The fields transforming as the trivial representation of D_2 are χ_1, χ_5, χ_9 and χ_{13}. Thus by the vacuum expectation values $\langle\phi(66)\rangle = \langle\phi(6-6)\rangle = v_1/\sqrt{2}$, $\langle\phi(64)\rangle = \langle\phi(6-4)\rangle = v_5/\sqrt{2}$, $\langle\phi(62)\rangle = \langle\phi(6-2)\rangle = v_9/\sqrt{2}$, and $\langle\phi(60)\rangle = v_{13}$ the SO(3) is broken to its finite subgroup D_2. With $w = 2\pi/3$ in $A'(w)$ of (39) we obtain χ_1, χ_8 and χ_{13} transforming as the trivial representation of D_3. Thus with the expectation values $\langle\chi_1\rangle = v_1$, $\langle\chi_8\rangle = v_8$, and $\langle\chi_{13}\rangle = v_{13}$, SO(3) breaks down to D_3. Another embedding of D_3 in SO(3) is possible if we replace B by $C = \exp \mathrm{i}\pi J_2$ in which case the trivials are χ_1, χ_7 and χ_{13}. The generators $A(2\pi/3)$, B, C together generate the dihedral group D_6 where χ_1 and χ_{13} serve as the trivial representation of D_6 so that $\langle\chi_1\rangle = v_1$ and $\langle\chi_{13}\rangle = v_{13}$ will break SO(3) to D_6. The dihedral group D_4 will be generated by B' and $A'(\pi/2)$ in which case χ_5 and χ_{13} are the trivial representations. The transformation properties of the other fields are also transparent. The generators A and C also generate the same dihedral group D_4.

Now, if we let $w = 2\pi/5$ in $A'(w)$ of (35) we observe that the scalars χ_4 and χ_{13} are the trivial representation of D_5 generated by $A(2\pi/5)$ and B. If we generate D_5 by $A(2\pi/5)$ and $C = \exp \mathrm{i}\pi J_2$ then the scalars χ_3 and χ_{13} transform as the trivial representation. Therefore, SO(3) can be broken to D_5 in two different ways. The matrices $A(2\pi/5)$, B and C generate the dihedral group D_{10} whose only trivial scalar is also the trivial scalar χ_{13} of D_∞. Therefore, D_{10} is not a little group for the largest group of this series D_∞ possesses the same trivial representation χ_{13}.

One could have generated D_6 by $A(2\pi/6)$ and B which also involves C as one of the element and the trivials are the same scalars χ_1 and χ_{13} which was discussed above.

The cyclic little groups are those $C_n (n = 1, 2, 3)$ for $j = 6$ where C_3, C_2 and C_1 have respectively five, seven and 13 trivial representations whose field contents can be identified from (38) and (39).

(b) *The icosahedral group*

We note from (5) that the 60 elements of the icosahedral group can be generated by $A = \exp \mathrm{i}(2\pi/5)Q$, $C = \mathrm{i}\pi J_3$ where $Q = (\sigma J_1 + J_3)/(\sqrt{2+\sigma})$ and J_i's are the 13×13 generators of the SO(3) Lie algebra, obtained from (1) for $j = 6$. The matrix C is a diagonal matrix with ± 1 entries whereas A is a complicated unitary matrix with entries of irrational numbers. The choice of the basis (38) transform these matrices to orthogonal matrices though not in the desired block diagonal form. It can be checked that the 13-dimensional irreducible representation of SO(3) can be decomposed as

$$\underline{13} = \underline{1} + \underline{3} + \underline{4} + \underline{5} \tag{40}$$

in terms of the irreducible representations of the icosahedral group Y. It is not, in principle, difficult to bring simultaneously the matrices A and C into block diagonal form but this needs a lot more computer calculations. A choice of the new basis $\eta = S\chi$ transforms A

and C into block diagonal forms

$$A' = \begin{bmatrix} 1 & & & \\ & L & & \\ & & M & \\ & & & N \end{bmatrix}$$

$$C' = \begin{bmatrix} 1 &&&&&&&&&&&& \\ & -1 &&&&&&&&&&& \\ && -1 &&&&&&&&&& \\ &&& 1 &&&&&&&&& \\ &&&& 1 &&&&&&&& \\ &&&&& -1 &&&&&&& \\ &&&&&& 1 &&&&&& \\ &&&&&&& -1 &&&&& \\ &&&&&&&& 1 &&&& \\ &&&&&&&&& 1 &&& \\ &&&&&&&&&& 1 && \\ &&&&&&&&&&& -1 & \\ &&&&&&&&&&&& -1 \end{bmatrix} \tag{41}$$

where L, M, N are 3×3, 4×4 and 5×5 orthogonal matrices, respectively. They are given in the appendix together with the representation contents of the η fields. Since the generators A' and C' satisfy the generation relation $A'^5 = B'^3 = C'^2$, with $C' = A'B'$ one can compute the generator B' from the relation $B' = A'^T C'$. One can easily show that $B'^3 = 1$.

From the appendix we note that the field

$$\eta_1 = \frac{1}{32\sqrt{6}}[6\sqrt{35}\chi_1 - 2\sqrt{462}\chi_5 - 6\sqrt{77}\chi_9 + 2\sqrt{66}\chi_{13}] \tag{42}$$

transforms as a trivial representation of the icosahedral group. The fields (η_2, η_3, η_4) transform as a triplet of Y much as the (W_1, W_2, W_3) do. The other fields

$$(\eta_5, \eta_6, \eta_7, \eta_8) \qquad \text{and} \qquad (\eta_9, \eta_{10}, \eta_{11}, \eta_{12}, \eta_{13})$$

transform respectively as four-and five-dimensional irreducible representations of Y. When the fields (η_2, η_3, η_4) are gauged away by an SO(3) transformation giving the non-zero vacuum expectation value to η_1, $\langle\eta_1\rangle = v_1$, the symmetry is broken to the icosahedral group. The original fields which gain the expectation values proportional to $\langle\eta_1\rangle = v_1$ are found to be

$$\begin{aligned} \langle\phi(66)\rangle &= \langle\phi(6-6)\rangle = \tfrac{\sqrt{3\cdot5\cdot7}}{32}v_1 \\ \langle\phi(64)\rangle &= \langle\phi(6-4)\rangle = -\tfrac{\sqrt{2\cdot7\cdot11}}{32}v_1 \\ \langle\phi(62)\rangle &= \langle\phi(6-2)\rangle = -\tfrac{\sqrt{3\cdot7\cdot11}}{32}v_1 \\ \langle\phi(60)\rangle &= \tfrac{\sqrt{11}}{16}v_1. \end{aligned} \tag{43}$$

The other fields take zero expectation values. All the gauge bosons gain equal masses given by $m_W = \sqrt{14}v_1$.

Before concluding this section we would like to note the little groups T and O of $j = 6$. With respect to the irreducible representations of T the 13-dimensional irreducible representation of SO(3) can be decomposed as

$$\underline{13} = 2(\underline{1}) + \underline{1}' + \underline{1}'' + 3(\underline{3}) \tag{44}$$

which indicates that T possesses two trivial representations in $j = 6$. Since T is a subgroup of Y one of its trivial representation is characterized by the field η_1. The second trivial scalar field is some linear combination of the χ fields which we did not attempt to identify. Similarly the $j = 6$ representation can be decomposed as

$$\underline{13} = \underline{1} + \underline{1}' + \underline{2} + 2(\underline{3}) + \underline{3}' \tag{45}$$

in terms of the irreducible representations of the octahedral group O. For the octahedral groups not a subgroup of Y the trivial representation in (45) is not the field η_1 and should be computed. This needs some further work. Nevertheless, under the light of these discussions, it is clear that the octahedral group is a little group of the $j = 6$ representation. The gauge bosons transform as the representation $\underline{3}$ so that the scalar fields belonging to the irreducible representation $\underline{3}$ can be gauged away and the gauge bosons gain equal masses.

6. Discussions

Using the matrix representations of the irreducible representations $j = 2, 3, 4, 6$ of SO(3) in the canonical basis we have obtained the matrix representations of the corresponding little groups. They are transformed into the block-diagonal forms so that the representation contents of the Higgs scalars turned out to be manifest. Assigning the vacuum expectation values to the Higgs scalars of the trivial representations of the related little groups, the SO(3) symmetry is broken to its closed subgroups. Three gauge bosons of SO(3) gain masses except in the case of the groups C_∞, C_n and D_∞, where W_3 remain massless.

What we have not discussed in the text is the problem of pseudo-Goldstone bosons emerging in these breakings; some general remarks can be made. The number of pseudo-Goldstone bosons can be predicted in each individual case for which the numbers of Higgs fields in the trivial representations and the massive gauge bosons are known. For example, when SO(3) $\rightarrow D_2$ occurs as a spontaneous breaking with the Higgs fields in the $j = 2$ representation no pseudo-Goldstone bosons arises since we have two Higgs scalars in the trivial representations and the remaining Higgs fields are gauged away to yield the longitudinal degrees of freedom to the W bosons. For the other little groups this is not the case and the number of pseudo-Goldstone bosons equals $2(j - 1)$ minus the number of trivial representations of the little groups for all cases except for the fact that it is $(2j - 1)$ minus the number of trivial representations for the groups C_∞, C_n and D_∞.

If SO(3) is embedded in a larger local symmetry and this larger symmetry is broken to the closed subgroups of SO(3), it is then possible that these pseudo-Goldstone bosons could be absorbed by the additional gauge bosons of the larger symmetry. If any closed subgroup of SO(3) is going to be a residual symmetry in some kind of GUT breaking, SO(3) symmetry has to be a component of the larger symmetry to avoid the pseudo-Goldstone bosons. The spontaneous breaking of GUT, with or without horizontal symmetry, to a theory with a residual finite subgroup of SO(3) induces the cosmic strings which can be characterized by the conjugacy classes of the finite subgroups of SU(2). Although there are a number of interesting works in the literature [6], this program requires more detailed analysis and is deferred for a further study.

Analogous structures, in the case of liquid crystals, have been suggested where the line defects are associated with the conjugacy classes of the binary polyhedral groups [3, 4]. Similar structures are expected in the phase transitions of the early universe where a GUT breaking may involve a closed subgroup of SO(3). In the light of the foregoing discussions our work constitutes a mathematical framework to implement such studies both in the field of liquid crystals and/or in cosmic strings.

A connection between the present work and the ADE series of the affine Lie algebras can be made where the McKay correspondence may play a fundamental role. A partial success has already been achieved [10] but definitely needs further investigations. It is also desirable to study the correspondence between the present method of symmetry breaking of SO(3) and the one made by tensor fields, which seems to be more appropriate in the liquid crystal phenomena. A detailed study of the symmetry breaking patterns for the $j \geqslant 3$ representation including the relations between the $\phi(jm)$ fields and the higher rank symmetric tensor fields $T_{abc\cdots}$ will be discussed elsewhere [14].

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Appendix

The matrices used in (41) are as follows:

$$L = \frac{1}{2}\begin{bmatrix} -\tau & \sigma & 1 \\ -\sigma & 1 & \tau \\ -1 & \tau & \sigma \end{bmatrix} \qquad \begin{aligned} \tau &= \tfrac{1}{2}(1+\sqrt{5}) \\ \sigma &= \tfrac{1}{2}(1-\sqrt{5}) \end{aligned} \tag{46}$$

$$M = \frac{1}{2}\begin{bmatrix} -1/3 & \sqrt{3} & \sqrt{5}/3 & -1/\sqrt{3} \\ -1/\sqrt{3} & -1 & \sqrt{5/3} & -1 \\ -2\sqrt{5}/3 & 0 & 1/3 & \sqrt{5/3} \\ -2/\sqrt{3} & 0 & -\sqrt{5/3} & -1 \end{bmatrix} \tag{47}$$

$$N = \frac{1}{8}\begin{bmatrix} 1 & 3-\sqrt{5} & -\sqrt{5} & 3+\sqrt{5} & -2\sqrt{5} \\ -3+\sqrt{5} & -4 & -\sqrt{3}(1+\sqrt{5}) & 0 & 4 \\ -\sqrt{15} & \sqrt{3}(1+\sqrt{5}) & -1 & \sqrt{3}(-1+\sqrt{5}) & 2\sqrt{3} \\ 3+\sqrt{5} & 0 & \sqrt{3}(-1+\sqrt{5}) & 4 & 4 \\ 2\sqrt{5} & 4 & -2\sqrt{3} & -4 & 0 \end{bmatrix}. \tag{48}$$

The irreducible representations $\underline{1}$, $\underline{3}$, $\underline{4}$, $\underline{5}$ of the icosahedral group and the η fields are

$$\underline{1}:\ \eta_1 = \tfrac{1}{32\sqrt{6}}[6\sqrt{35}\chi_1 - 2\sqrt{462}\chi_5 - 6\sqrt{77}\chi_9 + 2\sqrt{66}\chi_{13}] \tag{49}$$

$$\underline{3}: \begin{cases} \eta_2 = \tfrac{1}{32\sqrt{6}}[2\sqrt{3(83-33\sqrt{5})}\chi_3 + 6\sqrt{11(7+3\sqrt{5})}\chi_7 + 6\sqrt{22(3-\sqrt{5})}\chi_{11}] \\ \eta_3 = \tfrac{1}{32\sqrt{6}}[-2\sqrt{3(83+33\sqrt{5})}\chi_4 - 6\sqrt{11(7-3\sqrt{5})}\chi_8 + 6\sqrt{22(3+\sqrt{5})}\chi_{12}] \\ \eta_4 = \tfrac{1}{32\sqrt{6}}[-18\sqrt{10}\chi_2 + 8\sqrt{33}\chi_6 + 6\sqrt{22}\chi_{10}] \end{cases} \tag{50}$$

$$\underline{4} = \begin{cases} \eta_5 = \frac{1}{32\sqrt{6}}[-5\sqrt{22}\chi_1 - 2\sqrt{110}\chi_2 - 14\sqrt{15}\chi_5 - 8\sqrt{3}\chi_6 + 11\sqrt{10}\chi_9 \\ \qquad - \sqrt{2}\chi_{10} + 2\sqrt{105}\chi_{13}] \\ \eta_6 = \frac{1}{32\sqrt{6}}[6\sqrt{11(3-\sqrt{5})}\chi_3 - 6\sqrt{11(3+\sqrt{5})}\chi_4 - 2\sqrt{3(23+3\sqrt{5})}\chi_7 \\ \qquad + 2\sqrt{3(23-3\sqrt{5})}\chi_8 + 2\sqrt{3}(3+5\sqrt{5})\chi_{11} + 2\sqrt{3}(3-5\sqrt{5})\chi_{12}] \\ \eta_7 = \frac{1}{32\sqrt{6}}[-\sqrt{110}\chi_1 + 10\sqrt{22}\chi_2 - 14\sqrt{3}\chi_5 + 8\sqrt{15}\chi_6 + 11\sqrt{2}\chi_9 \\ \qquad + 14\sqrt{10}\chi_{10} + 2\sqrt{21}\chi_{13}] \\ \eta_8 = \frac{1}{32\sqrt{6}}[-6\sqrt{11(3-\sqrt{5})}\chi_3 - 6\sqrt{11(3+\sqrt{5})}\chi_4 + 2\sqrt{3(23+3\sqrt{5})}\chi_7 \\ \qquad + 2\sqrt{3(23-3\sqrt{5})}\chi_8 - 2\sqrt{3}(3+5\sqrt{5})\chi_{11} - 2\sqrt{6(67-15\sqrt{5})}\chi_{12}] \end{cases} \tag{51}$$

$$\underline{5} = \begin{cases} \eta_9 = \frac{1}{32\sqrt{6}}[-\sqrt{33(47-21\sqrt{5})}\chi_1 + 3\sqrt{2(27+7\sqrt{5})}\chi_5 \\ \qquad - \sqrt{15(47-21\sqrt{5})}\chi_9 + 3\sqrt{14(27+7\sqrt{5})}\chi_{13}] \\ \eta_{10} = \frac{1}{32\sqrt{6}}[-2\sqrt{66}\chi_2 - 24\sqrt{5}\chi_6 + 10\sqrt{30}\chi_{10}] \\ \eta_{11} = \frac{1}{32\sqrt{6}}[-3\sqrt{11(27+7\sqrt{5})}\chi_1 - \sqrt{6(47-21\sqrt{5})}\chi_5 \\ \qquad - 3\sqrt{5(27+7\sqrt{5})}\chi_9 - \sqrt{42(47-21\sqrt{5})}\chi_{13}] \\ \eta_{12} = \frac{1}{32\sqrt{6}}[-6\sqrt{11(7+3\sqrt{5})}\chi_3 + 2\sqrt{15(47-21\sqrt{5})}\chi_7 \\ \qquad + 2\sqrt{6(23+3\sqrt{5})}\chi_{11}] \\ \eta_{13} = \frac{1}{32\sqrt{6}}[6\sqrt{11(7-3\sqrt{5})}\chi_4 + 2\sqrt{15(47+21\sqrt{5})}\chi_8 \\ \qquad + 2\sqrt{6(23-3\sqrt{5})}\chi_{12}]. \end{cases} \tag{52}$$

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